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GEOMETRICALLY NONLINEAR ANALYSIS OF THE APOLLO AFT HEAT SHIELD

by

R. H. Gallagher, and R. H. Mallett

REPORT NO. 7218-933005

DECEMBER 1966

Final Report

NASA Contract NAS 9-3528

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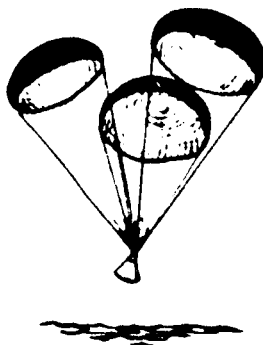
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FOREWORD

This report was prepared by Textron's Bell Aerosystems Company under NASA Contract NAS 9-3528 and covers work performed during the period 1 April 1966 to 15 December 1966. The contract was administered under the direction of Dr. F.J. Stebbins, Structures and Mechanics Division, at NASA, Manned Spacecraft Center.

Numerical results presented in this report were obtained, in part, with the digital computer facilities at the Manned Spacecraft Center.



ABSTRACT

The response of the Apollo heat shield structure is considered under water impact loading conditions. Equivalent static pressure loading is defined to transform the associated analysis problem to one of elastic stability. An analysis method is developed and geometrically nonlinear analyses are conducted to assess the significance of finite displacement effects.

The technical approach employed in both analytical formulation and numerical solution is one of a direct attack on the nonlinear problem. This approach enables the realistic utilization of a displacement criterion for the detection of elastic instability. Concepts and procedures embodied in the analysis method are fully explained in the context of the Apollo aft heat shield application.

Numerical results are presented and correlated with those reported in Reference 1 to illustrate clearly the effect of finite displacements. This report, together with the associated computer program, provides a basis for further analyses of the Apollo aft heat shield and many other NASA structures.



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I. INTRODUCTION

In September, 1964, the NASA Manned Spacecraft Center contracted with Bell Aerosystems Company for a six-month study of the application of the discrete element structural analysis method to the prediction of thermal stresses in the Apollo Command Module aft heat shield. Shortly thereafter, in December, 1964, the Apollo aft heat shield sustained damage during tank testing, occasioning efforts directed toward structural design changes, particularly with respect to the scalloping of the face sheets. Bell Aerosystems Company was asked to assist in this work, via an extension of the contract, and numerous analyses were performed to guide the specific nature of the scalloping to be effected. This latter effort, conducted within a very short period of time, was amplified in March, 1965, into a more extensive examination of the heat shield with respect to water impact and thermal stressing conditions. All aspects of this multiphase investigation were reported in Reference 1.

Both analytical and experimental determinations have given evidence that the stresses sustained in the Apollo aft heat shield, approach the level of the yield stress of the material under water impact loading conditions. Since the heat shield is a shallow, thin shell, it is important to confirm its integrity against elastic instability under these loadings.

The elastic instability analyses of the Apollo aft heat shield, conducted within the scope of Contract NAS 9-3528, were based on a generalization of well known beam-column concepts. The matrix methods of structural analysis, based on discrete element idealization, were extended to provide a suitable analytical and computational framework. This elastic instability analysis capability was evaluated by consideration of selected elementary example shell structures for which alternative analytical and experimental critical load predictions were available. The results of this evaluation, recorded in Reference 1, demonstrated good correlation of the matrix method predictions and classical solutions. The discrete element instability analysis method was then applied to the Apollo aft heat shield structure.

Three distinct water impact loadings were considered. The results, which were modified by means of empirical factors to account for finite displacement effects, indicated a small but positive margin of safety.

These critical load predictions were central to a comprehensive review of elastic instability phenomena pertinent to the Apollo aft heat shield which was undertaken jointly by consultants to NASA and personnel at NASA and Bell. As a consequence of this review, it was mutually agreed that an advanced instability analysis was warranted which would account for effects of finite displacements. The investigation reported herein was initiated to accomplish this purpose.

There exists a hierarchy of nonlinearities which may be retained in seeking to improve linearized analysis models. Associated with each level in this hierarchy is a level of complexity in formulation and solution as well as a class of problems for which the solution furnishes a realistic measure of actual behavior. As additional nonlinearities are considered, the pertinent problem class is expanded and complexity of formulation and solution is increased.

The nonlinearities pertinent to the subject shallow, thin shell problem are geometric nonlinearities; specifically, those associated with displacements normal to the midplane of the shell. The complexity associated with this level of nonlinearity has long defied the development of reliable working tools for finite displacement analyses. Classical methods of analysis, which have yielded tractable formulations for only a limited number of linear problems, are even less fruitful in the presence of geometric nonlinearities. Furthermore, nonlinear formulations successfully stated have commonly come to naught in consequence of the inadequacy of procedures for extracting numerical solutions. Computational procedures employed have generally exhibited unreliable and inefficient convergence characteristics. The nonlinear analysis method employed herein overcomes, in large measure, the difficulties which have plagued attempts to deal with this class of problem.

The approach is one of direct attack on the nonlinear problem by the retention of finite displacement terms in the strain-displacement relations. A basic need for load incrementation is thereby avoided. Broad applicability and simplicity of formulation are achieved by development within the framework of discrete element idealization.

Effective solution of this nonlinear problem is accomplished by recourse to a structural analysis method of recent origin (Reference 2). This method is based on the direct utilization of the principle of stationary potential energy, which states that (Reference 3):

Of all possible displacement states within a given admissible (see page 13) class $\{\Delta\}_{n \times 1}$, that which makes the potential energy Φ_p stationary, satisfies the equilibrium requirements and is the actual displacement state $\{\Delta\}^*$,

$$\left. \frac{\partial \Phi_p(\{\Delta\})}{\partial \Delta_j} \right|_{\{\Delta\} = \{\Delta\}^*} = 0 \text{ for all } j = 1, \dots, n \quad (1-1)$$

Furthermore, if

$$\Phi_p(\{\Delta\}^*) < \Phi_p(\{\Delta\}) \quad (1-2)$$

for all $\{\Delta\}$ in some neighborhood of $\{\Delta\}^*$, then the associated equilibrium position is stable.

This principle of potential energy provides the means for casting the subject geometrically nonlinear analysis as a problem in mathematical programming. The pertinent special case of the general mathematical programming problem, known as unconstrained minimization, is simply stated, (Reference 4)

$$\begin{aligned} \text{Given} \quad & F(\{x\}) \\ \text{find} \quad & \{x\}^* \\ \text{such that} \quad & F(\{x\}^*) = \text{Minimum} \end{aligned}$$

A necessary condition for the occurrence of a minimum is

$$\left. \frac{\partial F(\{x\})}{\partial x_j} \right|_{\{x\} = \{x\}^*} = 0 \quad \text{for all } j = 1, \dots, n$$

Determination of $\{x\}^*$ as the solution of a mathematical programming problem formed using the potential energy function is clearly equivalent to solving the direct displacement formulation (Equation 1-1).

The reasons for resorting to this alternative approach to solving the nonlinear set of equations of a direct formulation may not be obvious, but they are compelling. Most importantly, this alternative approach allows the powerful numerical methods of mathematical programming to be brought to bear on the geometrically nonlinear structural analysis problem posed by the Apollo aft heat shield under water impact loading conditions. These methods have been employed at Bell and elsewhere in dealing effectively with related nonlinear structural problems (References 5, 6, and 7).

The foregoing paragraphs have reviewed the events which motivated the subject investigation of finite displacement effects on the response of the Apollo aft heat shield under water impact loading and have outlined, conceptually, the technical approach employed. Analytical simplicity and broad applicability, the principal features of the linear matrix methods, have been preserved by constructing the nonlinear formulation as an extension of these methods. Statement of the resulting formulation as a mathematical programming problem has allowed advanced numerical methods to be brought to bear. Numerical results are reported which confirm the relative merit of this conceptual approach. Complete documentation is included herein for all phases of the investigation conducted.

Concepts and procedures pertinent to definition of a physical model for the Apollo aft heat shield are presented in Section II. Simulation of the heat shield structures by means of a shallow circular arch model is considered, as well as representation as an assembly of thin shell discrete elements.



Section II also presents in detail, the analytical development of appropriately nonlinear mathematical models. Specific reference is made to the frame discrete element in exhibiting algebraic detail. Explicit consideration is given to the combination of discrete elements to obtain a nonlinear mathematical model for the Apollo aft heat shield. Matrix notation is employed to facilitate relation to well known procedures for linear analysis.

Section III is devoted to a review of mathematical programming methods. The state-of-the-art is assessed and the basis for selecting the particular technique employed is explained. Results are presented and discussed in Section IV.

These results are interpreted to achieve the study objective of estimating the effects of finite displacements on the response of the Apollo aft heat shield under water impact loading conditions.

Section V is comprised of an in depth review of the investigation conducted. Major emergent conclusions and recommendations are listed in Section VI.

II. ANALYTICAL METHOD

A. INTRODUCTION

In accordance with the conceptual approach outlined in Chapter 1, the objective mathematical model for the Apollo aft heat shield takes the form of an expression for the total potential energy. Using discrete element concepts to advantage, individual element potential functions are constructed. The pertinent geometric nonlinearities are introduced into the element potential energies through the strain displacement relations. The total potential energy for the structure is then formed simply as the scalar sum of the element potential energies. The following paragraphs delineate the analytical procedure for the formulation of these potential energy functions.

Two structural discrete elements have been employed to model the Apollo aft heat shield. The first is a frame element for the idealization of shallow, circular arch representations of the heat shield structure. This element, shown in Figure 1, is assumed straight, slender, and free of stress in the undeformed structure. Deformation of the element is permitted to occur without deformation of the cross section. Cross-sections initially plane remain plane under element deformation. Double symmetry of the cross-section is assumed. Both membrane and flexure behavior are admitted.

The second structural discrete element is a triangular shell element. This element, shown in Figure 2, is a thin shell element of zero curvature which affords realistic polygonal approximation of the shallow heat shield structure. The element has an arbitrary triangular shape. The sandwich construction of the heat shield is accommodated by the assumption of distinct membrane and flexure material properties. The high span-to-thickness ratio of the heat shield eliminates the need to consider shear deformation.

It is pertinent to note that these discrete element physical models afford convenient consideration of variations in the heat shield structure. The ability to predict the behavior resulting from the scalloping of the face sheets and the existence of a varying pressure over only a portion of the surface are problematical in the context of classical analysis methods. In the subject method the ability to assign different properties to the individual elements easily accounts for these variations.

B. FRAME ELEMENT REPRESENTATION

Detailed presentation of the analysis capability described above is given in the following with specific reference to the frame element. The procedure is applicable without conceptual extension to more complex elements including the triangular thin shell element.

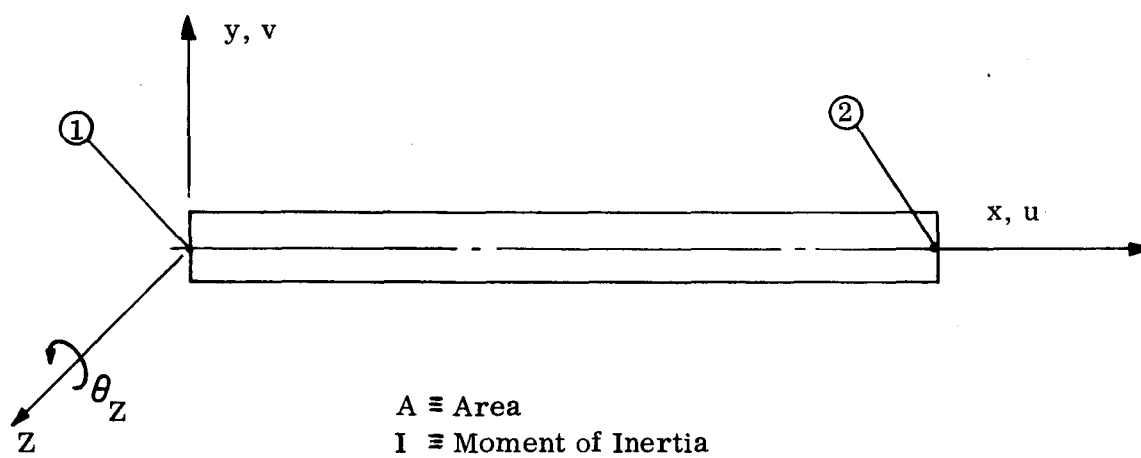


Figure 1. Frame Discrete Element

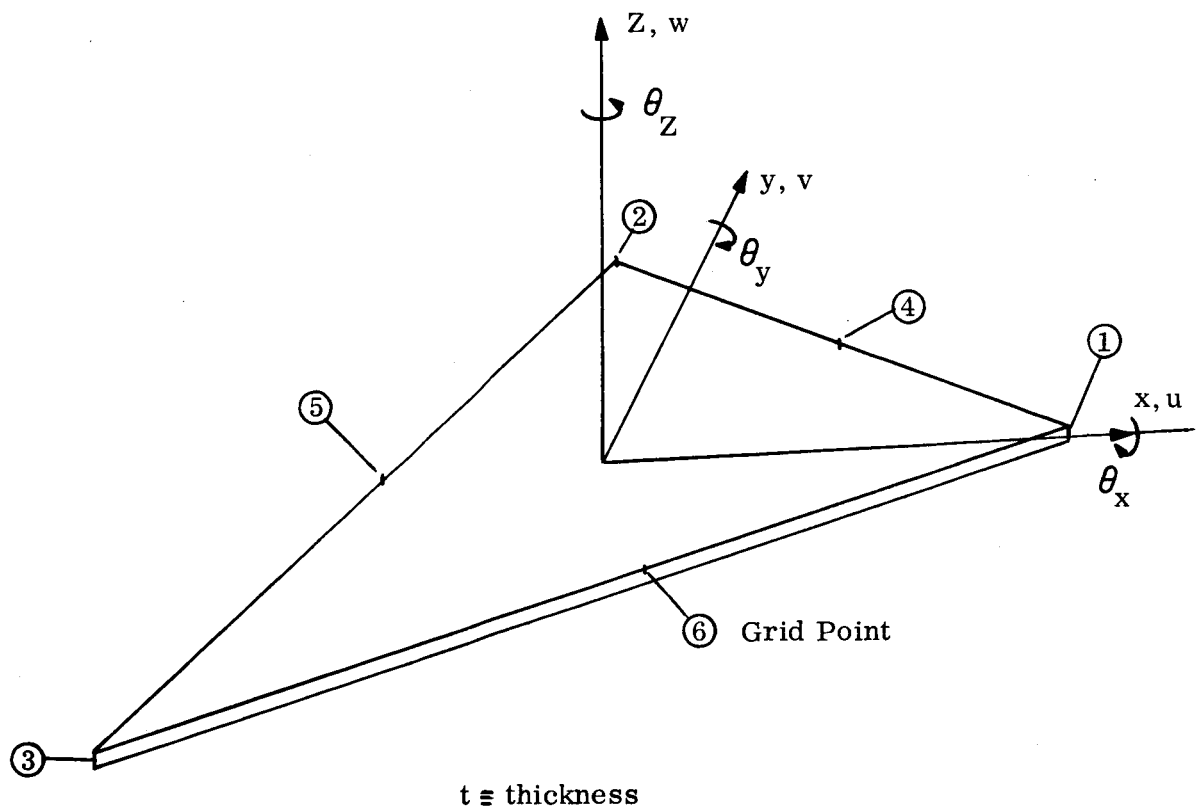


Figure 2. Shell Discrete Element



The fundamental requirements for the frame discrete element, and indeed all elasticity problems, are satisfied by establishing:

1. Equilibrium
2. Material behavior
3. Compatibility
4. Boundary conditions

Use of the previously stated principle of potential energy (Equations 1-1 and 1-2) foregoes the need for explicit consideration of equilibrium requirements since the equilibrium equations arise indirectly as the Euler equations in this variational approach. Furthermore, explicit consideration of boundary conditions is not necessary. Geometric boundary conditions are satisfied by the selection of admissible assumed displacement functions. The force boundary conditions are natural boundary conditions and are, therefore, satisfied automatically (Reference 3).

Explicit consideration must be given to writing equations governing stress-strain and strain-displacement behavior. Turning attention first to material behavior, linear elastic behavior governed Hooke's law is assumed; i.e.,

$$\sigma(x, y) = E \epsilon(x, y) \quad (2-1)$$

where

$$\sigma(x, y) = \text{stress}$$

$$\epsilon(x, y) = \text{strain}$$

$$E = \text{Modulus of elasticity}$$

The remaining fundamental requirement is satisfied explicitly by writing a strain-displacement relation; i.e.,

$$\epsilon(x, y) = u_x(x) + \frac{1}{2} v_x^2(x) - y v_{xx}(x) \quad (2-2)$$

where the subscript "x" indicates partial differentiation with respect to "x" and

$$u(x) = \text{membrane displacement}$$

$$v(x) = \text{flexure displacement}$$

It is this strain-displacement relation which accounts for finite displacement effects and correspondingly introduces non-linearity into the problem by the retention of the second order term v_x^2 . This term is not small relative to the linear terms in the presence of finite displacements.

The total potential energy may be written as the difference of the strain energy (U) and the external work (W).

$$\Phi_p = U - W \quad (2-3)$$

In virtue of the linearity of the material, (Equation 2-1), the strain energy is given by,

$$U = \int_V \frac{1}{2} E \epsilon^2 dV \quad (2-4)$$

The element strain energy functional follows by introduction of the strain-displacement relation, (Equation 2-2), and integration over the doubly symmetric cross section.

$$U(u(x), v(x)) = \int_0^L \left[\frac{EA}{2} u_x^2 + \frac{EI}{2} v_{xx}^2 + \frac{EA}{2} u_x v_x^2 + \frac{EA}{8} v_x^4 \right] dx \quad (2-5)$$

The first two energy contributions are recognizable as the well known membrane and flexure terms, respectively. The remaining two terms exist in consequence of the provision for finite displacements.

The first of these is the important nonlinear membrane-flexure coupling term. It is this contribution, as will be demonstrated subsequently, which gives rise to terms which adversely affect the element linear membrane and flexure stiffnesses. This stiffness degradation leads, with increasing load, to elastic instability.

The final energy term is a higher order bending contribution which serves to counteract the degradation of bending stiffness. This term may be neglected in the absence of significant bending prior to the occurrence of instability.

The next analytical step in proceeding toward a nonlinear element potential energy function is to effect a discretization of the functional by the assumption of displacement functions. For the frame element of reference the following polynomial shapes are assumed.

$$\begin{aligned} u(x) &= a_0 + a_1 x \\ v(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 \end{aligned} \quad (2-6)$$

These displacement functions, upon substitution into Equation 2-5 and integration, yield an algebraic expression for the element strain energy

$$U(\{\beta\}) = \frac{1}{2} [\beta] [\tilde{K}] \{\beta\} + \frac{1}{3} [\beta] [\tilde{N1}] \{\beta\} + \frac{1}{4} \beta [\tilde{N2}] \{\beta\} \quad (2-7)$$

where

$\{\beta\}$ is the vector of field coordinate displacement degrees of freedom

$$\{\beta\}^T = [a_0, a_1, b_0, b_2, b_3] \quad (2-8)$$

and the coefficient matrices are,

$$[\tilde{K}]: \quad \begin{aligned} \tilde{K}_{22} &= +EAL \\ \tilde{K}_{55} &= +4EIL \\ \tilde{K}_{65} &= \tilde{K}_{56} \end{aligned} \quad \begin{aligned} \tilde{K}_{56} &= +6EIL^2 \\ \tilde{K}_{66} &= +12EIL^3 \end{aligned}$$

$$\begin{aligned} \frac{1}{EA} [\tilde{N1}]: \quad \tilde{N1}_{24} &= \frac{1}{2} \beta_4 + \frac{L^2}{2} \beta_6, \tilde{N1}_{25} = \frac{L^2}{2} \beta_4 + \frac{3L^4}{4} \beta_6 + \frac{2L^3}{3} \beta_5 \\ \tilde{N1}_{26} &= \frac{L^3}{2} \beta_4 + \frac{3L^4}{4} \beta_5 + \frac{9L^5}{10} \beta_6, \tilde{N1}_{42} = \tilde{N1}_{24}, \tilde{N1}_{44} = \frac{L}{2} \beta_2, \tilde{N1}_{45} = \frac{L^2}{2} \beta_2 \\ \tilde{N1}_{46} &= \frac{L^3}{2} \beta_2, \tilde{N1}_{52} = \tilde{N1}_{25}, \tilde{N1}_{54} = \tilde{N1}_{45}, \tilde{N1}_{55} = \frac{2L^3}{3} \beta_2, \tilde{N1}_{56} = \frac{3L^4}{4} \beta_2 \\ \tilde{N1}_{62} &= \tilde{N1}_{26}, \tilde{N1}_{64} = \tilde{N1}_{46}, \tilde{N1}_{65} = \tilde{N1}_{56}, \tilde{N1}_{66} = \frac{9L^5}{10} \beta_2 \end{aligned} \quad (2-9)$$

$$\begin{aligned} \frac{1}{EA} [\tilde{N2}]: \quad \tilde{N2}_{44} &= \frac{L}{2} \beta_4^2 + \frac{2L^3}{3} \beta_5^2 + \frac{9L^5}{10} \beta_6^2 + L^2 \beta_4 \beta_5 + L^3 \beta_4 \beta_6 + \frac{3L^4}{2} \beta_5 \beta_6 \\ \tilde{N2}_{45} &= \frac{L^2}{2} \beta_4^2 + L^4 \beta_5^2 + \frac{3L^6}{2} \beta_6^2 + \frac{4L^3}{3} \beta_4 \beta_5 + \frac{3L^4}{2} \beta_4 \beta_6 + \frac{12L^5}{5} \beta_5 \beta_6 \\ \tilde{N2}_{46} &= \frac{L^3}{2} \beta_4^2 + \frac{6L^5}{5} \beta_5^2 + \frac{27L^7}{14} \beta_6^2 + \frac{3L^4}{2} \beta_4 \beta_5 + \frac{9L^5}{5} \beta_4 \beta_6 + 3L^6 \beta_5 \beta_6 \\ \tilde{N2}_{54} &= \tilde{N2}_{45} \end{aligned} \quad (2-10)$$

$$\tilde{N}_{55} = \frac{2L^3}{3} \beta_4^2 + \frac{8L^5}{5} \beta_5^2 + \frac{18L^7}{7} \beta_6^2 + 2L^4 \beta_4 \beta_5 + \frac{12L^5}{5} \beta_5 \beta_6 + 4L^6 \beta_5 \beta_6$$

$$\tilde{N}_{56} = \frac{3L^4}{4} \beta_4^2 + 2L^6 \beta_5^2 + \frac{27L^8}{8} \beta_6^2 + \frac{12L^5}{5} \beta_4 \beta_5 + 3L^6 \beta_4 \beta_6 + \frac{36L^7}{7} \beta_5 \beta_6$$

$$\tilde{N}_{64} = \tilde{N}_{46}, \tilde{N}_{65} = \tilde{N}_{56}$$

$$\tilde{N}_{66} = \frac{9L^5}{10} \beta_4^2 + \frac{18L^7}{7} \beta_5^2 + \frac{9L^4}{2} \beta_6^2 + 3L^6 \beta_4 \beta_5 + \frac{27L^7}{7} \beta_4 \beta_6 + \frac{27L^8}{4} \beta_5 \beta_6$$

This statement of the element strain energy is referenced to the field coordinate displacement degrees of freedom $\{\beta\}$ which arose as undetermined coefficients of the polynomial-assumed displacement functions. In order to enable interconnection of adjacent structural elements, it is necessary to transform to physical displacement degrees of freedom defined at the boundaries (end points) i.e.,

$$\begin{Bmatrix} u(0) \\ v(0) \\ v_x(0) \\ u(1) \\ v(1) \\ v_x(1) \end{Bmatrix} = \begin{bmatrix} 1 & & & & & \\ & & 1 & & & \\ & & & 1 & & \\ 1 & L & & & & \\ & & 1 & L & L^2 & L^3 \\ & & & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{Bmatrix} \quad (2-11)$$

This relationship is analytically inverted to yield the desired transformation between field and boundary displacement degrees of freedom.

$$\{\beta\} = \begin{bmatrix} \Gamma_{\beta\delta} \end{bmatrix} \{\delta\} \quad (2-12)$$

where

$$\{\delta\}^T = \begin{bmatrix} u_1, v_1, \theta_{z1}, u_2, v_2, \theta_{z2} \end{bmatrix} \quad (2-13)$$

$$[\Gamma_{\beta\delta}] = \begin{bmatrix} +1 & , & , & , & , & , \\ +\frac{1}{L} & , & , & , & +\frac{1}{L} & , \\ , & +1 & , & , & , & , \\ , & , & +1 & , & , & , \\ , & +\frac{3}{L^2} & , & +\frac{2}{L} & , & +\frac{3}{L^2} & , & +\frac{1}{L} \\ +\frac{2}{L^3} & , & +\frac{1}{L^2} & , & , & -\frac{2}{L^3} & , & +\frac{1}{L^2} \end{bmatrix} \quad (2-14)$$

Application of this transformation yields the desired form of the element potential energy mathematical model.

$$\Phi_p = \frac{1}{2} [\delta] [K] \{\delta\} + \frac{1}{3} [\delta] [N1] \{\delta\} + \frac{1}{4} [\delta] [N2] \{\delta\} - [\delta] \{P\} \quad (2-15)$$

where

$$\begin{aligned} \{P\}^T &= [P_{x1}, P_{y1}, M_{z1}, P_{x2}, P_{yz}, M_{z2}] \\ [K] &= [\Gamma_{\beta\delta}] [\tilde{K}] [\Gamma_{\beta\delta}] \\ [N1] &= [\Gamma_{\beta\delta}]^T [\tilde{N1}] [\Gamma_{\beta\delta}] \\ [N2] &= [\Gamma_{\beta\delta}]^T [\tilde{N2}] [\Gamma_{\beta\delta}] \end{aligned} \quad (2-16)$$

The gradient to this potential energy function is required to effect the numerical minimization efficiently. Recognizing that the matrices $[N1]$ and $[N2]$ are, respectively, linear and quadratic functions of the independent variables, the gradient takes the form,

$$\{\nabla \Phi_p\} = \left[[K] + [N1] + [N2] \right] \{\delta\} - \{P\} \quad (2-17)$$

This completes specification of the mathematical model for the slender prismatic element which is used to idealize shallow arch representations of the Apollo aft heat shield. The matrix $[K]$ can be recognized as the well known linear stiffness contribution. The matrix $[N1]$, referred to as the first order incremental stiffness, accounts for nonlinear membrane-flexure coupling. The effect is a mutual degradation of flexure stiffness by membrane behavior and membrane stiffness by flexure behavior. This is the mechanism by which instability occurs. In the context of linear algebra this is interpreted as an eroding of the positive definiteness of the stiffness matrix $[K]$.



At the point of elastic instability the modified stiffness becomes non-positive definite and the stability test of Equation 1-2 fails. This point also defines the limit of applicability of the solution procedure.

The matrix $[N_2]$, referred to as the second order incremental stiffness matrix, represents a higher order stiffening effect of flexure on flexure. This term is important to problem classes in which significant bending occurs prior to instability and in all applications where post buckling positions are sought.

C. SHELL ELEMENT REPRESENTATION

The triangular thin shell structural element generalized for nonlinear analysis of the Apollo aft heat shield is illustrated in Figure 2. The flexible triangular shape of this element enables representation of thin shell structures of arbitrary configuration; thereby, extending applicability of the subject analysis capability to a broad class of NASA structures.

The unique conceptual advancement embodied in this triangular shell element resides in the utilization of assumed displacement functions which satisfy admissibility conditions with respect to the total structure. The admissibility of assumed displacement functions requires that they be complete up to the order of truncation, embody all rigid body modes, and provide for interelement continuity (Reference 8). The satisfaction of admissibility guarantees that the predicted total potential energy will be an algebraic upper bound on the potential energy of the exact solution. Furthermore, the predicted potential energy will monotonically approach the exact value with grid refinement, theoretically converging to the exact value in the limit. Grid refinement is taken to mean the addition of grid points by further subdivision of the structure.

A concise historical review of the development of discrete element concepts is included in Chapter V of Reference 1. As pointed out therein, the definition of general systematic procedures for constructing approximate displacement functions within the confines of admissibility requirements has proved to be an elusive goal. Thus, it is particularly noteworthy that these admissibility conditions are satisfied by the assumed displacement functions in the subject triangular thin shell element.

As shown in Figure 2, six boundary gridpoints are defined on the triangular element. The use of six grid points to describe membrane deformation, rather than the usual three grid points, provides for variation in strain within the element. The accuracy of the element is thereby greatly enhanced relative to the constant-strain, three-grid point element. Flexure considerations at the midpoints are limited to normal slope degrees of freedom. The addition of these degrees of freedom refines the representation of deformation and enables satisfaction of interelement continuity requirements.

The strain energy function which serves as a basis for the triangular thin shell element may be written as

$$U = \int_V = \frac{1}{2} [\epsilon()] [E] \{\epsilon()\} dv \quad (2-18)$$

where

$$\{\epsilon()\}^T = [\epsilon_x, \epsilon_y, \epsilon_{xy}]$$

$$[E] = \text{coefficient matrix of generalized Hooke's law.}$$

The pertinent nonlinear strain displacement relations are given by (Reference 9):

$$\begin{aligned} \epsilon_x &= u_x + \frac{1}{2} w_x^2 - zw_{xx} \\ \epsilon_y &= v_y + \frac{1}{2} w_y^2 - zw_{yy} \end{aligned} \quad (2-19)$$

$$\epsilon_{xy} = u_y + v_x + w_x w_y - 2zw_{xy}$$

Formulation of the nonlinear potential energy mathematical model for the thin shell element proceeds in direct analogy with the procedure carried out in detail for the slender prismatic element. This algebraically complex development is presented in Reference 10. The result is documented in the computer program delivered under the present effort.

D. TOTAL STRUCTURE REPRESENTATION

The foregoing developments have dealt exclusively with the individual discrete elements. The remaining consideration is one of assembling the individual elements to form a representation of the total structure. The two items required by the numerical algorithm discussed in Section III, are the value of the total potential energy and its gradient.

The first item, the total potential energy of the structure for any specified displacement state, is simply the scalar sum of the individual element energy values. Calculation of the gradient to the potential energy function at any given point is also straightforward. It is constructed by effecting a nonconformable sum of element gradient vectors into the corresponding vector for the total structure. Regarding the potential energy gradient assembly process, it is understood that transformations have been applied to element degrees of freedom to obtain a consistent notation among elements common to a grid point.



The mathematical model for the nonlinear analysis of the Apollo aft heat shield, based on construction of the potential energy function, has numerous recommending features. Its foundation in the matrix methods based on discrete element idealization has preserved the broad applicability characteristic of these methods. Conceptual simplicity is achieved by the direct incorporation of nonlinearities arising in the strain-displacement relations. Finally, the approach is one which is amenable to numerical solution.

III. COMPUTATIONAL METHOD

A. INTRODUCTION

In Section I, the principle of minimum potential energy was employed to cast the general elasticity problem into the form of a mathematical programming problem:

Given $\Phi_p(\{\delta\})$

find $\{\delta\}^*$

such that

$$\Phi_p(\{\delta\}^*) = \text{Minimum}$$

In Section II, a suitably nonlinear mathematical model for the Apollo aft heat shield was cast in the form of a potential energy function $\Phi_p(\{\delta\})$. This Section deals with the numerical prediction of behavior by the application of a minimization technique to seek the stationary point $\{\delta\}^*$ which is known to describe the actual displacement state. Description of the specific minimization technique incorporated in the present investigation is prefaced by a concise review of alternative approaches in order to justify the selection made.

B. MATH PROGRAMMING CONCEPTS

A function minimization technique is an algorithm for choosing test points $\{\delta\}$ which provide information about the function $\Phi_p(\{\delta\})$ and the location of its minimum $\{\delta\}^*$. Minimization techniques are divisible into two general classes, sequential and nonsequential, according to the approach taken in the selection of test points. In a nonsequential search a complete set of test points is chosen prior to the initiation of testing. The function is evaluated at each test point and the lowest value obtained is taken to be the minimum of the function. A nonsequential minimization technique is, then, basically a strategy for selecting a suitable set of test points. One approach is to choose the test points at random according to an n-dimensional probability density function (Reference 11).

The probability density function is generally chosen as flat in the absence of information about the nature of the function. A second approach is to choose test points in a specific geometric pattern such as n-dimensional gridwork. This is called the factorial technique (Reference 12).

Attempts have been made to place the selection of a set of test points on a mathematical rather than an intuitive or random basis (Reference 13). The applicability of these nonsequential minimization techniques is generally restricted to situations in which the physical system defies mathematical representation. Even in these situations the tendency is to adopt a sequential staging of the nonsequential phases in order to reduce the number of test points required (Reference 14).

Sequential search techniques are appropriate for the continuous and differentiable analytic functions considered herein. The sequential search techniques seek to move from a given point $\{\delta\}_q$ a distance t_q along a direction $\{\phi\}_q$

$$\{\delta\}_{q+1} = \{\delta\}_q + t_q \{\phi\}_q \quad (3-1)$$

such that the function value is reduced.

$$\Phi_p(\{\delta\}_{q+1}) < \Phi_p(\{\delta\}_q) \quad (3-2)$$

With reference to, (Equation 3-1), the unconstrained minimization problem is reduced to determining which way to go ($\{\phi\}_q$) and how far to go (t_q) in the modification of the current point ($\{\delta\}_q$) so as to obtain a solution ($\{\delta\}^*$) with minimum effort. The selection of the optimum minimization technique is dependent upon the nature of the function (Reference 15).

The negative gradient direction was originally proposed by Cauchy as the answer to the "which way to go" question (Reference 16). This approach has come to be known as the method of steepest descents or the method of optimum gradients depending upon the criterion employed in determining "how far to go". In the method of optimum gradients the step length is determined so as to minimize the function along the gradient direction while in the method of steepest descents it is required only that some improvement in the function be achieved. Both methods have been found inefficient to the point of being useless for solving large nonlinear problems. The inefficiency is characterized geometrically by zig-zag behavior caused by the eccentricity of the function contours (Figure 3). A number of schemes have been put forward in attempting to overcome the zig-zag problem. Weighting and scale factors are possible for functions having certain special properties (References 17 and 18).

The periodic introduction of different types of moves has proved fairly successful (References 19 and 20). The most effective gradient techniques are based on the idea of conjugate directions. A conjugate direction method recently introduced by Fletcher (Reference 21), is probably the most powerful analytic gradient procedure now known for finding a local minimum of a general function. Relaxation minimization techniques result from the use of univariate modifications as an alternative answer to "which way to go".

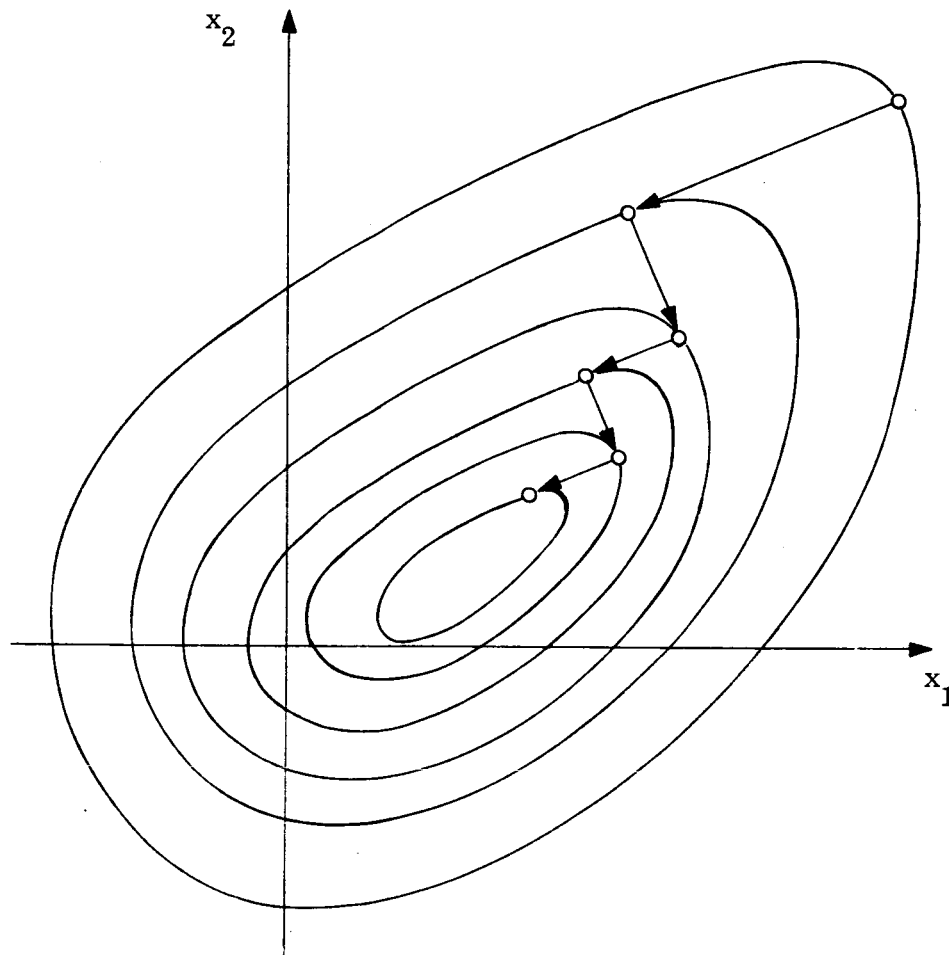


Figure 3. Zigzag in Two Dimensions



A number of ways have been suggested for choosing the coordinate to be modified in seeking to improve the current point. The increase in speed of convergence may not compensate for the time spent in the selection of the "best" univariate direction. Some methods of selecting a "best" univariate direction which have been proposed require a computational investment comparable with that necessary to determine the gradient direction (Reference 22). A more complete discussion of these methods can be found in Reference 23.

An approach to minimization which combines some of the aspects of the univariate and gradient methods has been reported (References 24 and 25). In this method sets of n univariate moves are alternated with moves called "pattern" moves. A pattern move consists of an extrapolation along the line defined by the initial and final points of the previous set of univariate moves. This direction clearly tends toward the minimum of the function. The method combines the simplicity of the univariate search with the larger steps which can be obtained using a gradient method. This approach to minimization is particularly recommended for situations in which an analytic gradient is not available or the computer storage capacity is limited. Otherwise, the best gradient methods are likely to be more efficient.

The question of "how far to go" is a one-dimensional optimization problem which exists without regard to how the "which way to go" question has been answered. The efficiency of any minimization procedure depends critically on the one-dimensional search technique employed. In some methods, such as in the method of steepest descents, the problem is avoided. However, in most methods an attempt is made to minimize the function along the chosen direction. A frequently used technique is the Fibonacci method (Reference 26). This method approximates the optimum along the given line by assuming the optimum to be within some initial interval called the "initial interval of uncertainty". This interval is then made arbitrarily small by systematically evaluating the function within the interval and excluding portions of the interval.

The most widely used technique for dealing with this problem is to fit a low order polynomial through a set of test points. The minimum of the actual function is assumed to coincide with the minimum of the approximating polynomial. This interpolation may be repeated over decreasing intervals to refine the approximation.

C. FR METHOD

The specific technique selected for incorporation herein was developed by Fletcher and Reeves, (FR method), in Reference 27. In this approach the powerful technique of Reference 21, is modified to eliminate the high computer program storage requirement of a square metric matrix. The basic quadratic convergence characteristic of Reference 21 is preserved.



Quadratic convergence means that, in the absence of round-off errors, the minimum of a quadratic function will be located in n steps where n is the number of variables. This is important in the context of more general functions since these become increasingly more closely quadratic as the minimum is approached.

The FR method also retains the basic conjugate directions concept which serves to eliminate the zig-zag convergence characteristics of the steepest descent methods. Numerical results are presented in Reference 27 for the FR method. Convergence is shown, for two frequently used test problems, to require about twice the cycles of the Fletcher and Powell method.

The FR method was selected in preference to the method of Reference 21, which embodies a square metric matrix for two reasons. Firstly, provision of storage for the square metric matrix would have greatly handicapped efficient structuring of the overall computer program. Secondly, it was anticipated that the convergence speed advantage of the square metric method would be dissipated by the associated matrix manipulation in large order problems.

It is pertinent to note here that, based on experience in numerical applications, it is subsequently recommended, (Section VI) that the highly directed method of Reference 21 be applied within the framework of the larger MSC computing facility to obtain more efficient operation.

IV. APPLICATIONS

A. INTRODUCTION

This chapter describes the utilization of the analysis capability developed to assess the effect of finite displacements on the response of the Apollo aft heat shield under water impact loading conditions. The problems considered may be divided into two classes; namely, those useful for confirming proper operation of the computer program and those directly pertinent to the Apollo aft heat shield.

B. APPLICATIONS

The first test application is illustrated in Figure 4. The structure is a deep circular arch. The arch has a span of 34 inches and a rise of 9.818 inches. Motion normal to the plane of the arch is prevented. The ends of the arch are completely fixed against translation and rotation. A uniform pressure loading is applied normal to the circular mid-plane over the entire length of the arch. The uniform cross section of the slender arch, is characterized by an area "A" and a bending moment of inertia "I".

This test application was also employed in the previous phase of the present contract to confirm proper operation of the more restricted elastic instability analysis capability. Such results are documented in Reference 1. In order to facilitate correlation with these results, the same polygonal idealization of the arch is employed for the present test application.

The behavior predicted for this problem is exhibited in Figure 6 in the form of a plot of pressure intensity versus midspan displacement. The critical loads predicted by the method of Reference 1 and by classical method, (Equation 28) are exhibited on the load-displacement curve. These critical load values are seen to correspond to the maximum load level sustained prior to snap through using the subject finite displacement analysis designated the MIN- Φ_P method. Additionally, this correlation of results confirms the validity of neglecting finite displacement effects in investigating the stability of deep arches and shells.

The second application is illustrated in Figure 5. The structure is a shallow circular arch which differs from the first test application in that the rise is reduced to 1.090 inches. This shallow arch configuration is subjected to a vertical concentrated load at midspan.

As before, the predicted behavior is exhibited as a plot of applied load intensity versus midspan displacement (Figure 7). The critical load predicted by the method of

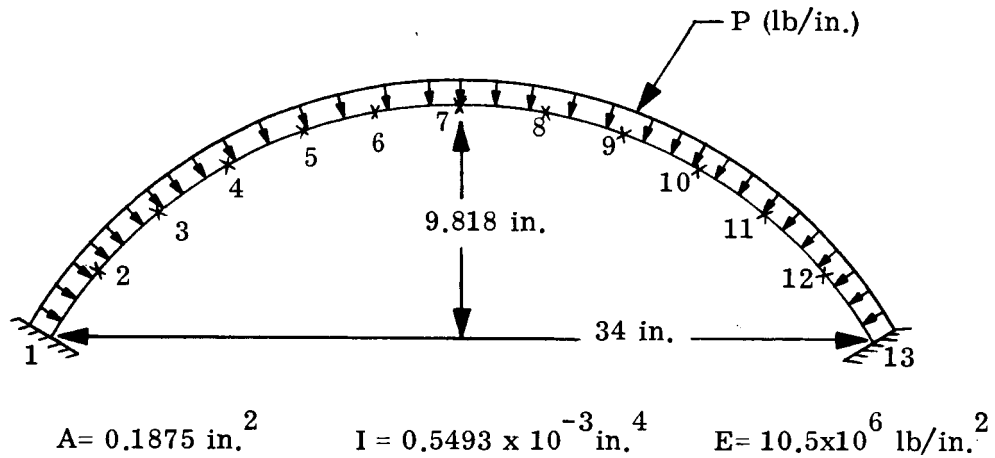


Figure 4. Case TA-1: Deep Circular Arch

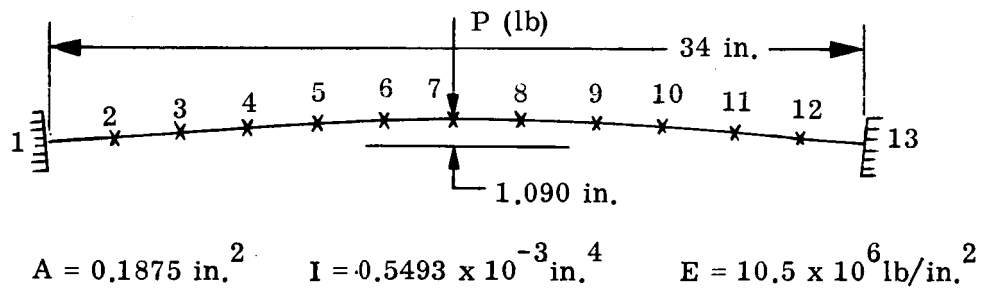


Figure 5. Case TA-2: Shallow Circular Arch

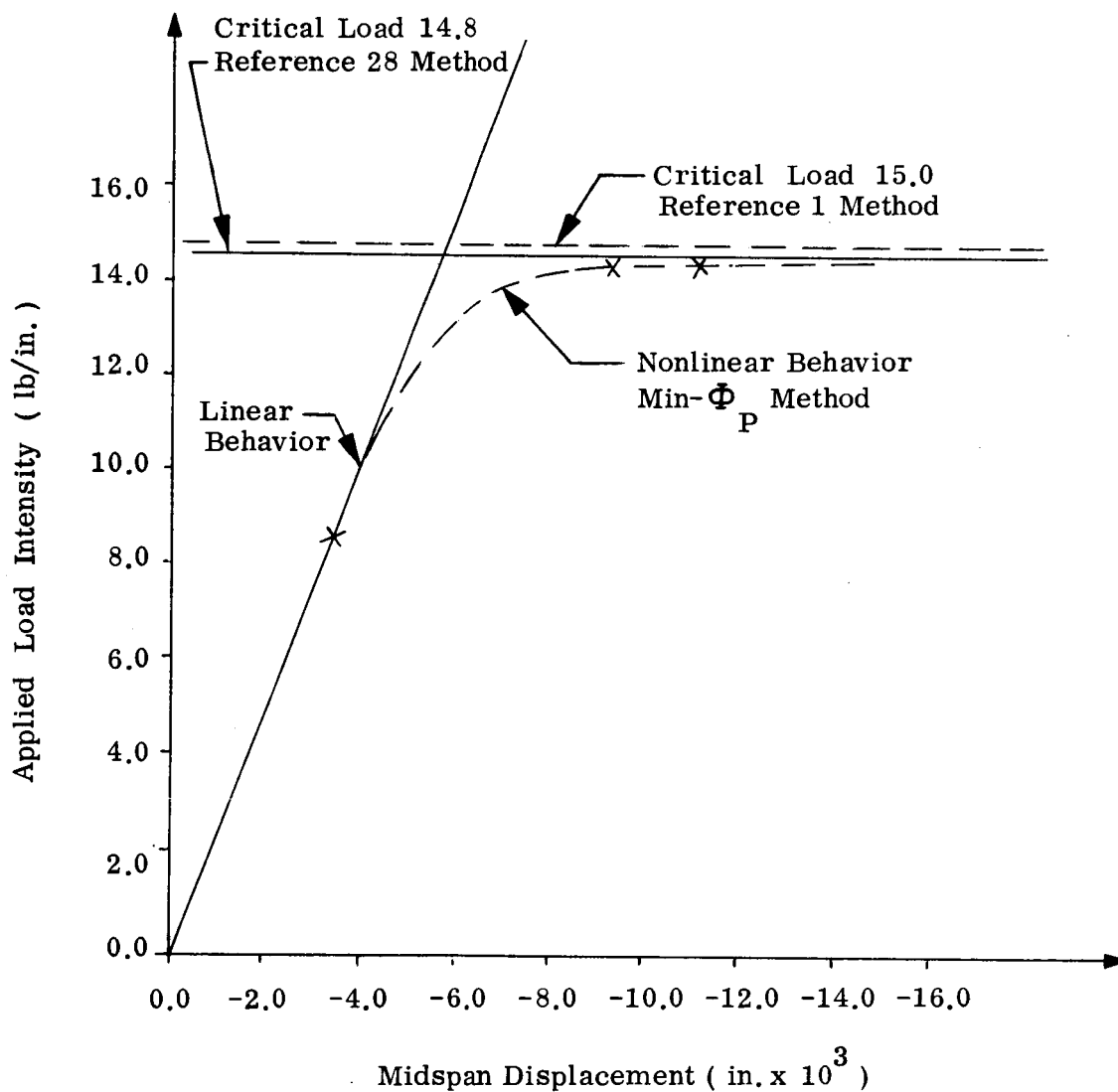


Figure 6. Case TA-1: Deep Circular Arch Behavior

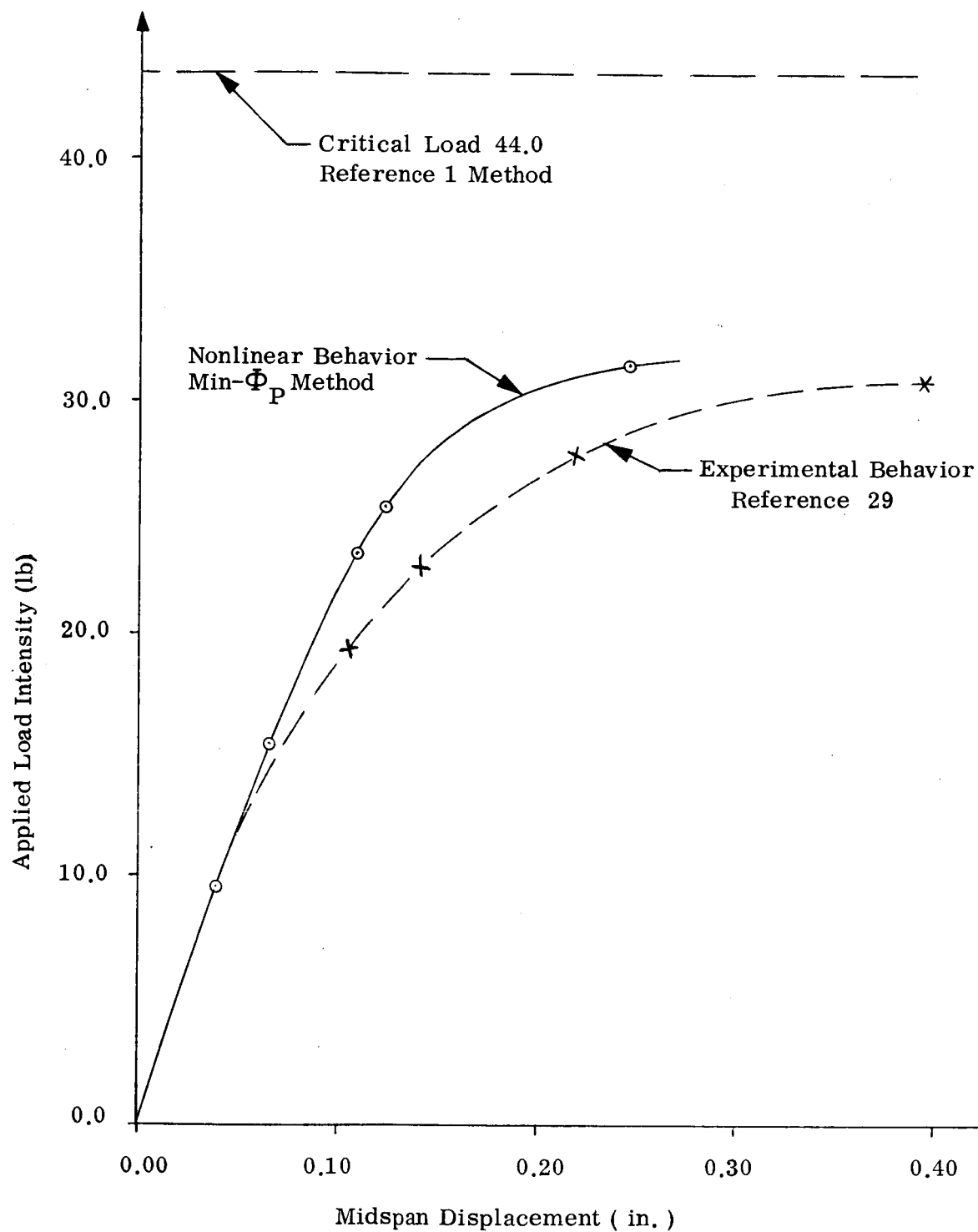


Figure 7. Case TA-2: Shallow Circular Arch Behavior



Reference 1 and an experimental plot of load displacement behavior Reference 29 are also shown. The correlation between the predicted finite displacement behavior and the experimental results confirms the proper operation of the analysis capability for this non-linear class of problems. The more pronounced knee in the theoretical result is characteristic of such comparisons with experimental data.

This example problem is of special interest since it makes evident the refinement introduced by the consideration of finite displacements in shallow arch and shell applications. The critical load P_{cr} predicted using the method of Reference 1 was 44 lb. The maximum load P_{max} sustained according to the subject MIN- Φ_p analysis was 32 lb. This variance, which stems from finite displacement effects, can be used to form a reduction factor C_r :

$$C_r = \frac{P_{max}}{P_{cr}} = 0.73$$

It is imprudent to utilize this reduction factor quantitatively for a shell structure; however, it is instructive to note that the arch has a rise to span ratio of only 0.03. A lower empirical reduction factor of 0.50 was applied in Reference 1 in prescribing the maximum permissible load level for the Apollo aft heat shield structure which has a higher rise to span ratio of 0.08.

The third application is illustrated in Figure 8. This structure is a shallow spherical shell similar to the Apollo aft heat shield. The spherical shell has a rise of 0.413 inch and a span of 8.000 inches. The edge of the shell is completely fixed against translation and rotation. The uniform thickness of the shell is 0.0537 inch. The material is isotropic with a modulus of elasticity of 6.5×10^6 psi and a Poisson ratio of 0.32. A uniform pressure is prescribed over the entire shell.

This application was also considered in Reference 1. The idealization of the shell, shown in Figure 8 is more refined than that employed in Reference 1. In virtue of the axi-symmetry of the structure and loading only a quadrant of the complete shell structure is considered.

A linear analysis was conducted for a pressure intensity of 20 psi. The predicted normal displacement behavior along a radial line is exhibited in Figure 9. Superimposed on the same figure are the displacement profile presented in Reference 1, and a displacement profile obtained experimentally (Reference 30). These results confirm the proper operation of the advanced triangular shell discrete element.

The next application is that of the Apollo aft heat shield structure itself. The idealization employed is illustrated in Figure 10. The dimensions employed are those specified in detail in Reference 1. Linear analyses were conducted under

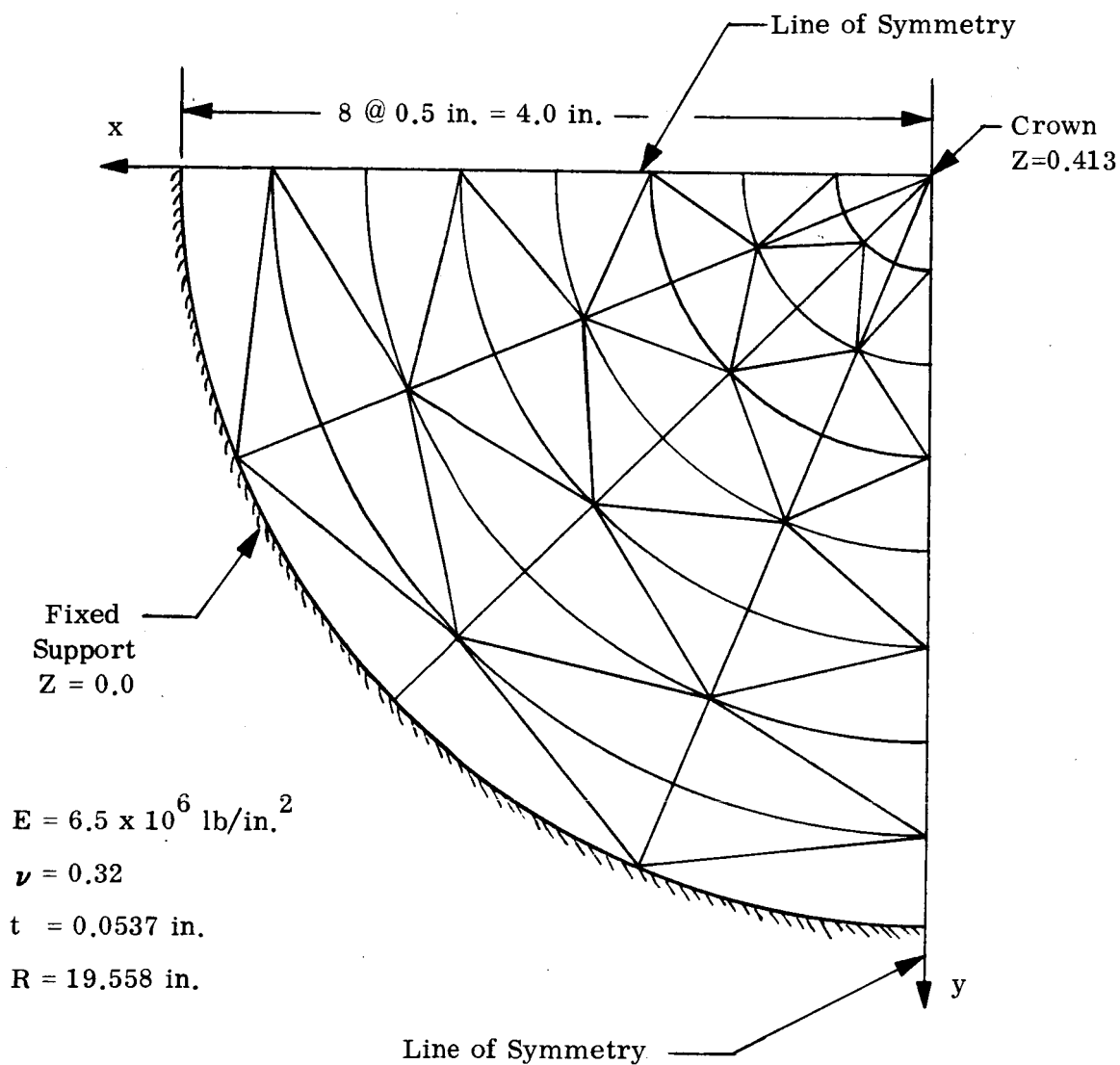


Figure 8. Case TA-3: Shallow Spherical Shell Description

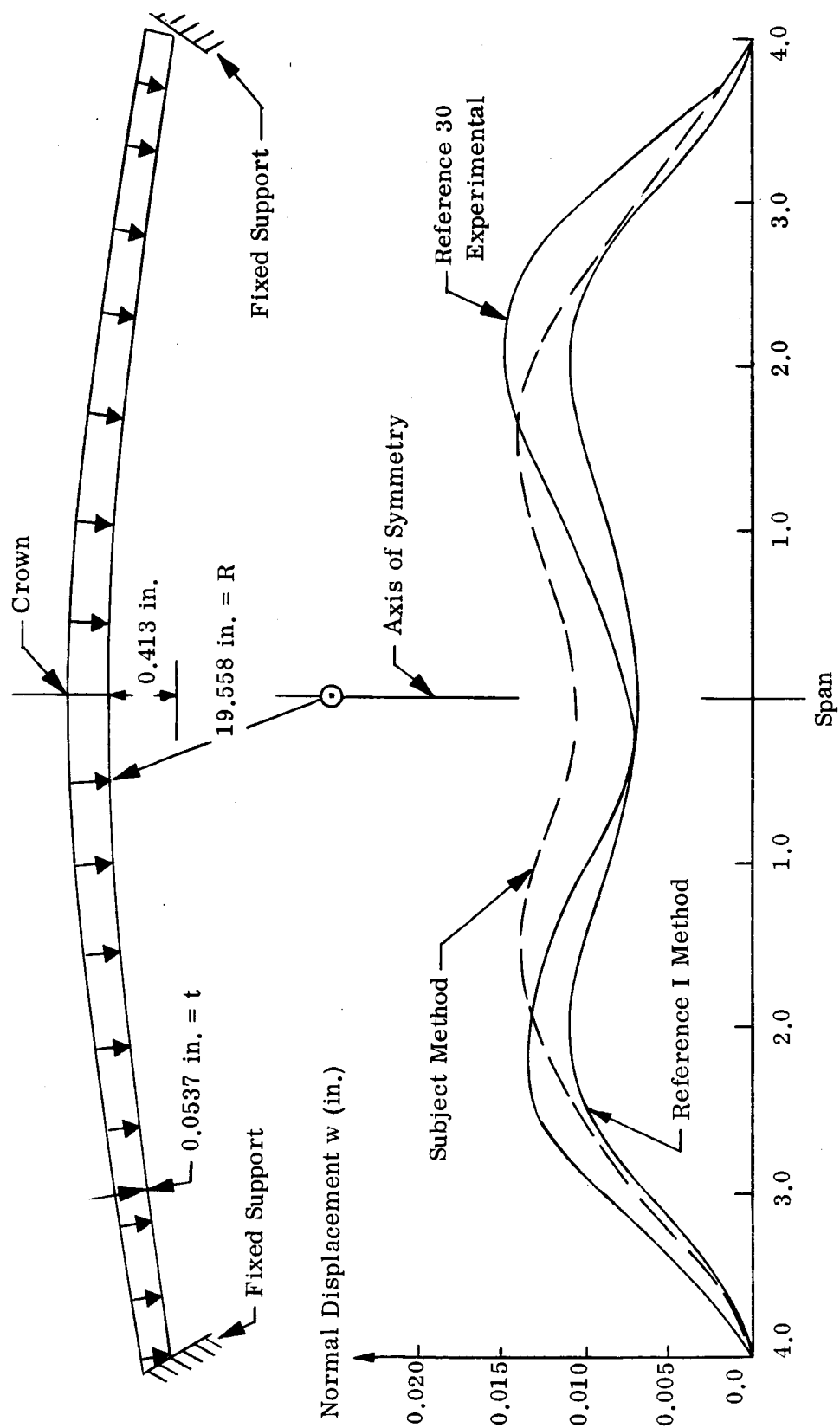


Figure 9. Case TA-3: Shallow Spherical Shell Behavior

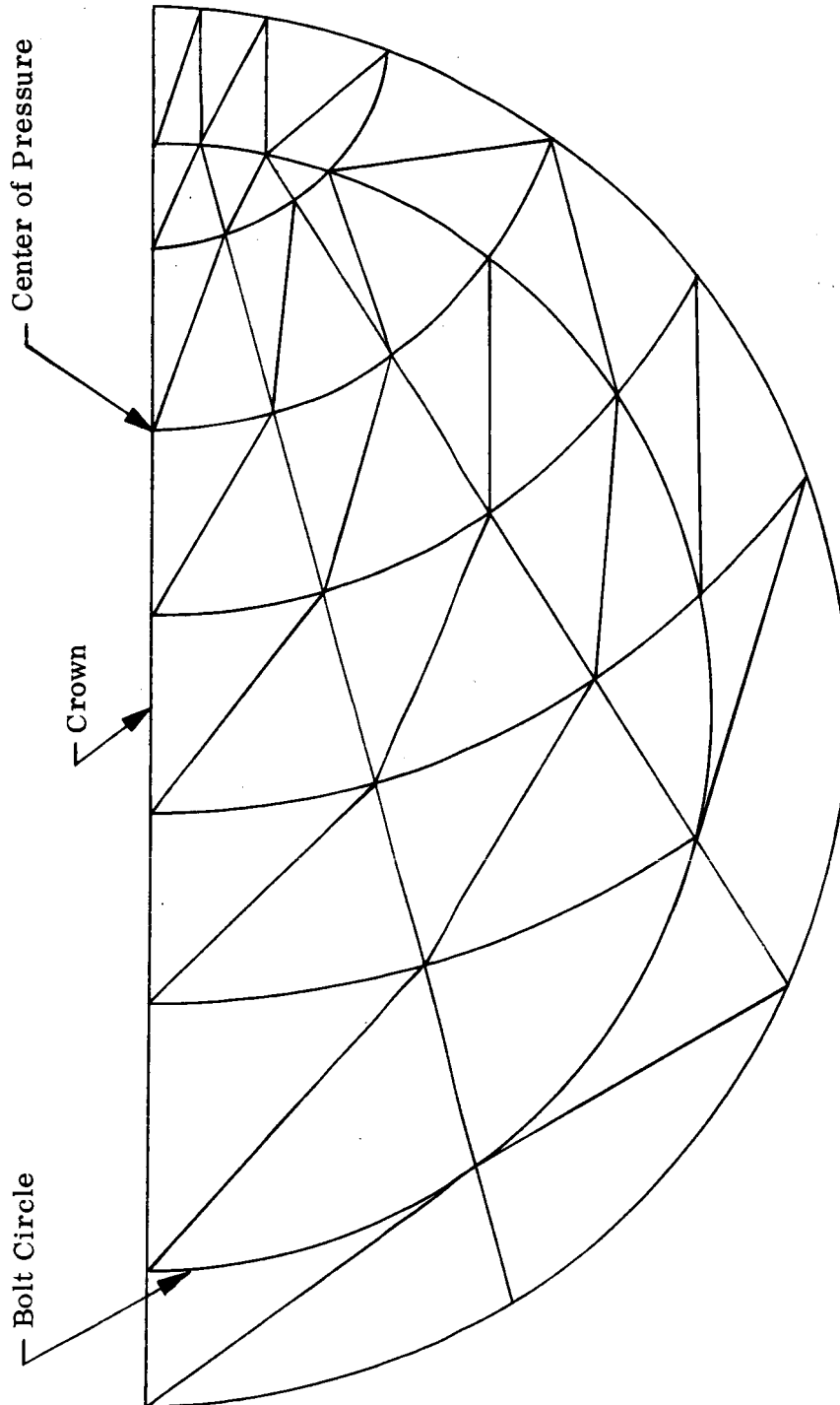


Figure 10 Cases AA-4,5,6: Apollo Heat Shield Shell Model Description



water impact loading for radii of pressure area equal to 10, 20, and 40 inches. The impact angle was taken to be 10 degrees. These loading conditions were determined to be the most critical in the stability investigation reported in Reference 1.

The behavior predicted for the Apollo aft heat shield under this water impact loading is illustrated in, Figures 11, 12 and 13, as plots of normal displacements induced along the great circle in the plane of symmetry. These results, obtained with the subject advanced ^{an}triangular shell element, are superposed on the corresponding results reported in Reference 1. The good correlation, ^{which is} evident confirms the validity of the predictions.

Even the slight differences in predicted behavior can be rationally attributed to variance in idealization. A relatively coarse grid was employed for the present analyses. This tends to concentrate the pressure loading causing greater displacement in the region of the loading. This greater displacement together with the greater stiffness of the coarse idealization combine to undercut the outward displacement in the region removed from the load.

The foregoing applications, taken collectively, serve to validate the geometrically nonlinear analysis capability developed for the Apollo aft heat shield.

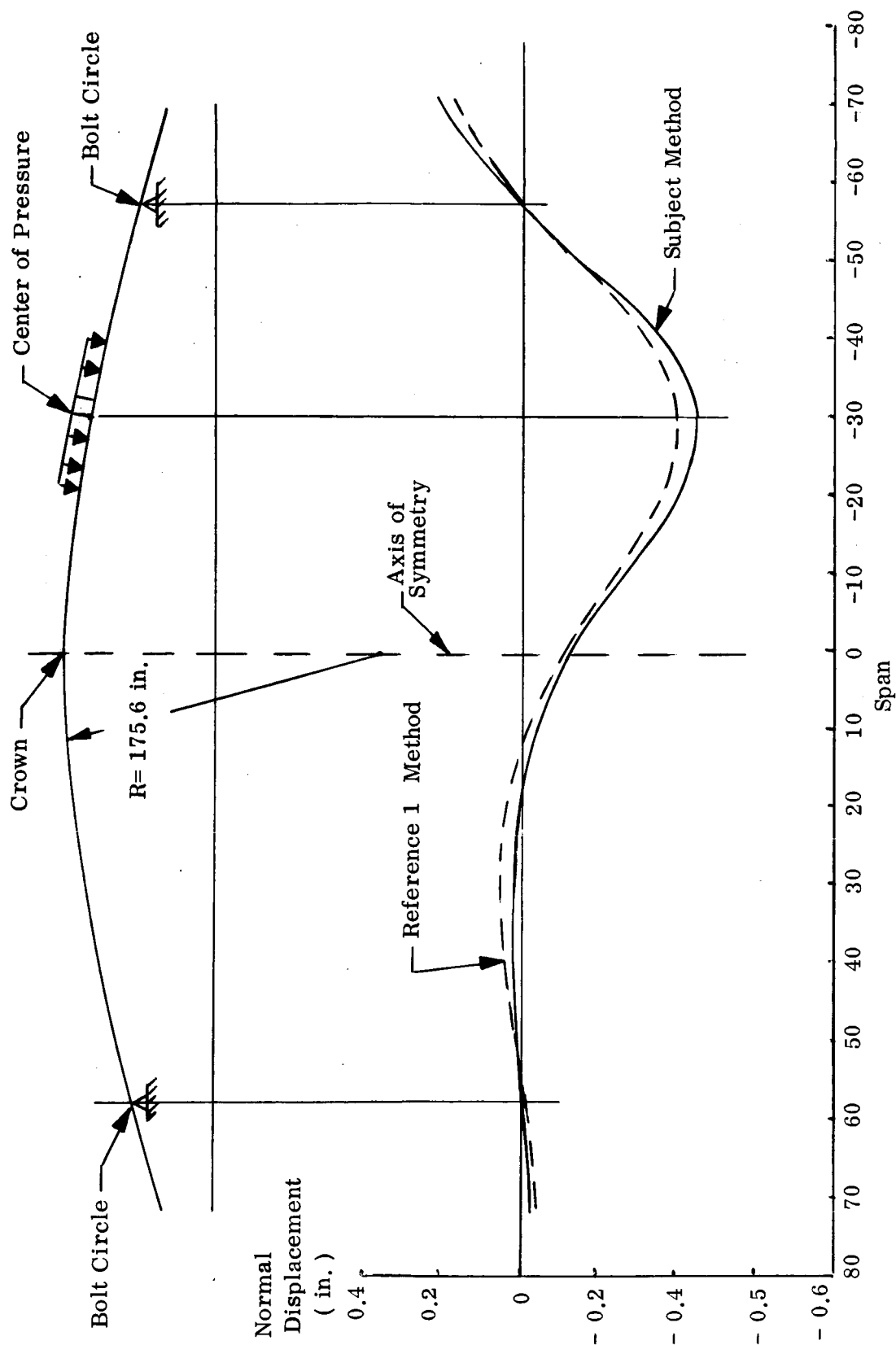


Figure 11. Case AA-4: Apollo Aft Heat Shield Behavior
10 in. Radius Area of Pressure
10 Deg. Angle of Impact

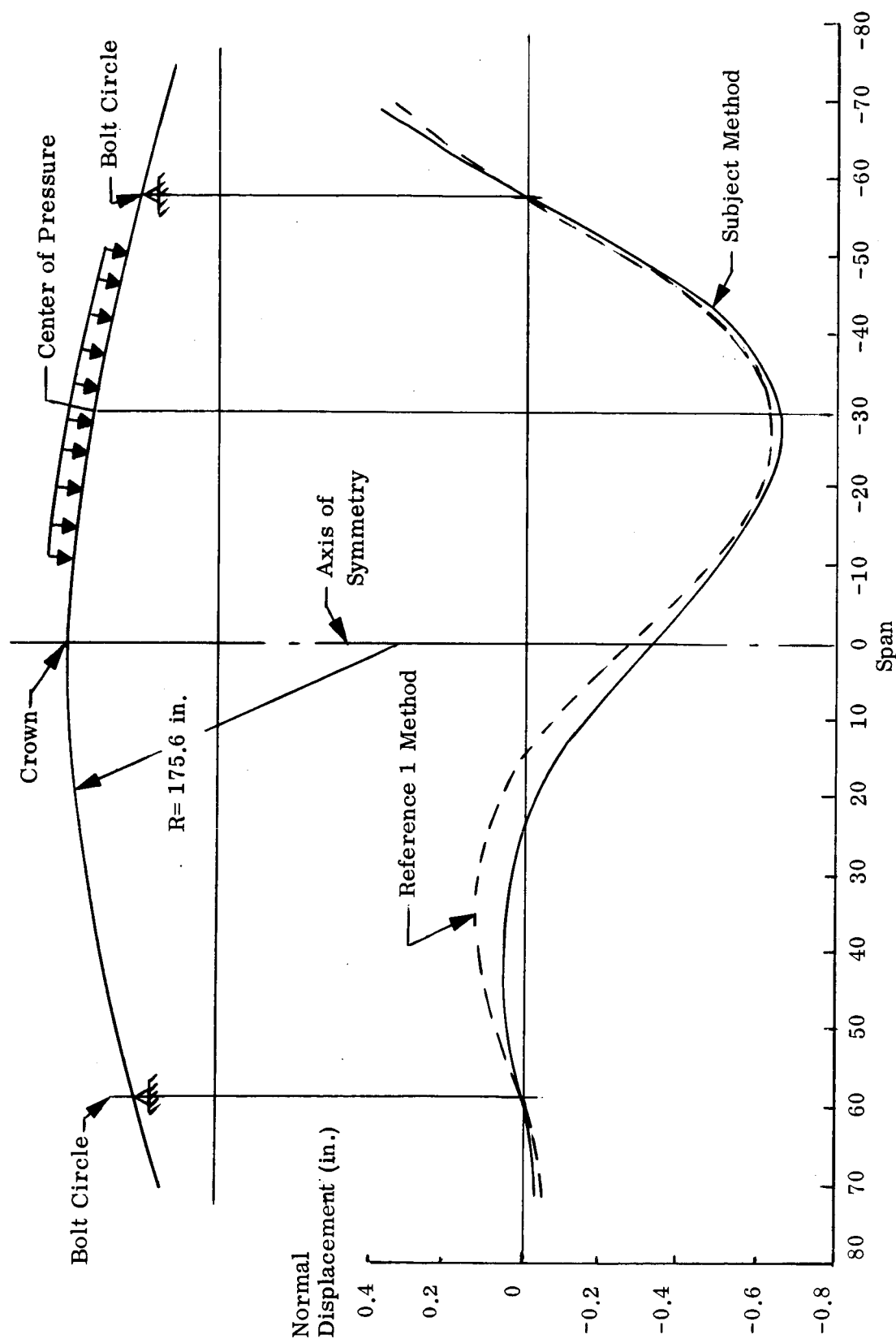


Figure 12. Case AA-5: Apollo Aft Heat Shield Behavior
20 in. Radius Area of Pressure
10 Deg. Angle of Impact

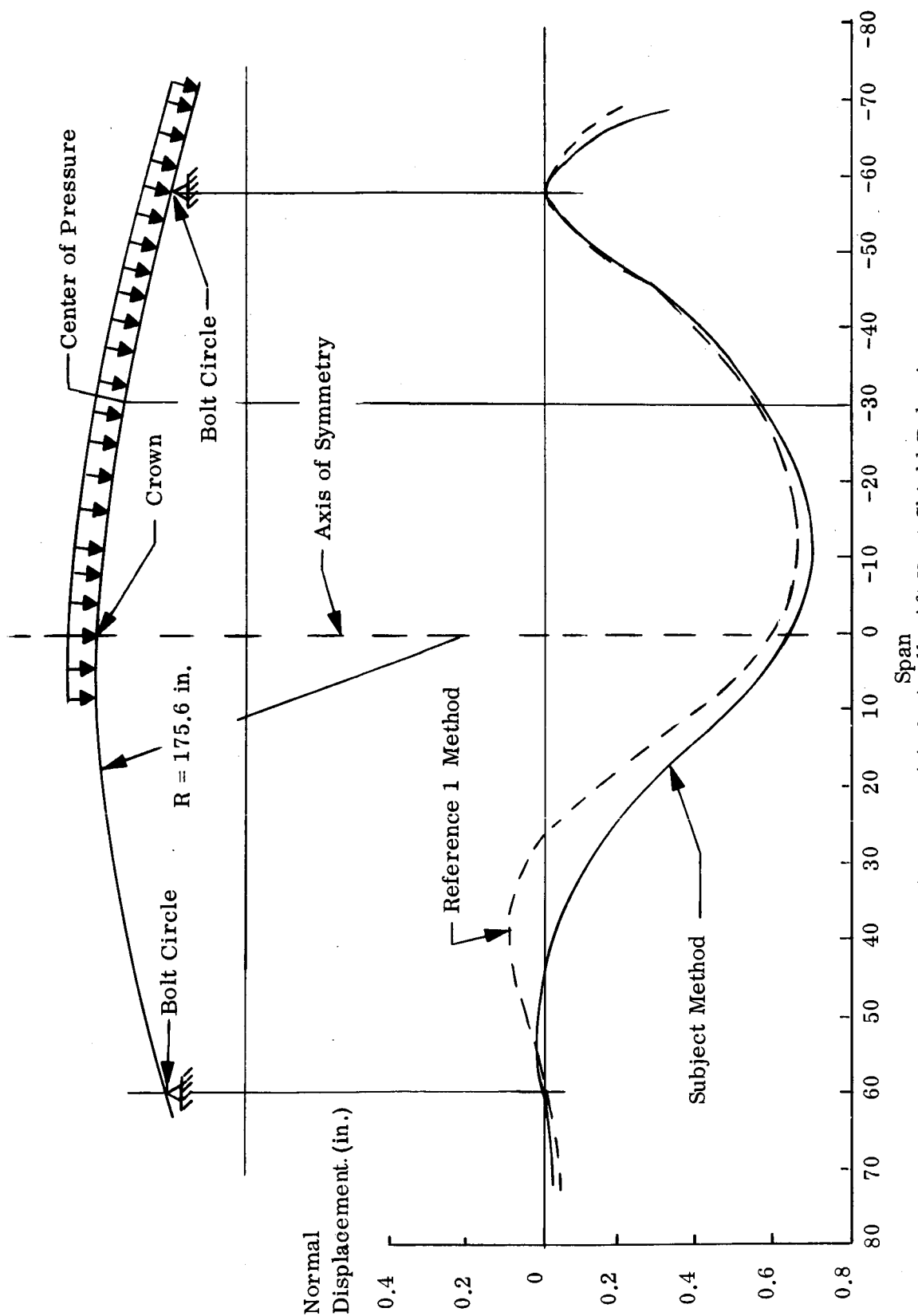


Figure 13. Case AA-6: Apollo Aft Heat Shield Behavior
40 in. Radius Area of Pressure
10 Deg. Angle of Impact



V. DISCUSSION

The primary objective of this investigation was to assess the effect of finite displacements on the response of the Apollo aft heat shield under water impact loading conditions. The successful accomplishment of this objective is manifest in the presentation and interpretation of results in Section IV, and in the delivery of a computer program for conducting further numerical analyses. These items provide the basis for confirming the integrity of the Apollo aft heat shield structure under water impact which was predicted in Reference 1.

The formulation and solution of any significant nonlinear structural analysis problem is exceedingly difficult. The complexities are largely inherent in the physical phenomena of elastic instability and are not subject to removal by ingenious methods of approach. However, the approach taken herein is thought to possess several important recommending features. Not the least of these is a compatibility with the NASA general purpose computer program for structural analysis which is currently under development (Reference 31). With this perspective, the following paragraphs are devoted to articulate retrospective examination of the subject nonlinear analysis capability.

This investigation embodied several associated research tasks, namely, the derivation of advanced nonlinear discrete element representations, the design of nonlinear structural analysis methods, and the evaluation of numerical solution techniques. The integrated results of these efforts were incorporated in a computer program to produce a working tool for the analysis of nonlinear structures.

With regard to discrete element representations, it is the conviction of the authors that utilization of the basic concept of discrete element idealization is invulnerable for the same reasons that have dictated general utilization of this concept in linear structural analysis. However, the assumptions invoked in definition, the form employed in statement, and the procedures followed in utilization require reexamination in the context of nonlinear problems.

The assumptions invoked in establishing the fundamental requirements of elasticity for a structural discrete element are brought into special focus by the implied requirement for high precision associated with a nonlinear analysis. Cognizance of this need in the present study is reflected in the sophisticated displacement modes employed for discretization of the triangular thin shell element. On the basis of these modes, the triangular thin shell element uniquely satisfies admissibility conditions established within the framework of the approximate variational methods of continuum mechanics.

As a consequence, recourse can be had to the convergence criteria stated in Section II for the interpretation of predicted behavior. Improvement beyond the subject element will stem from the elimination of idealization error by the use of curved elements. A further, more obvious, impact of the reexamination of basic assumptions is the retention of nonlinear terms in the strain-displacement relations which govern element deformation. These nonlinear terms are fundamental to detection of elastic instability phenomena.

Matrix notation has been the form employed for statement of linear mathematical models for discrete elements. The highly organized abstraction of discrete element methods of structural analysis results from the use of matrix notation. In effecting generalization to nonlinear formulations, however, matrix notation is unnatural and cumbersome. It does have the advantage of allowing convenient interpretation of nonlinear terms in the context of the familiar matrix statement of the linear problem and, for this reason, was employed in this study. It is anticipated that the importance of this consideration will diminish as experience with nonlinear problems is accumulated and that a concise summation notation which is amenable to computer programming will emerge as the preferable notation.

The final items mentioned here for special consideration regarding nonlinear discrete element representations was procedures followed in their utilization. The algebraic complexity inherent in effecting explicit analytical integration of the nonlinear terms for an element is formidable. This situation is aggravated by the trend toward discrete elements which eliminate discretization and idealization errors. The effect of this increasing algebraic complexity is to motivate automated integration procedures. Analytic and numerical alternatives are available. Numerical integration is expedient; however, automated analytic integration is generally more efficient. Thus, automated analytic integration was used to advantage in the triangular thin shell element representation.

The second task of study was identified as the definition of nonlinear structural analysis methods. Here, reference is made to the problem of stating the governing system of nonlinear algebraic equations in a form amenable to solution. The form resulting from the stationary condition of the total potential energy for a structure is:

$$[K] \{\Delta\} + [N_1(\Delta)] \{\Delta\} + [N_2(\Delta^2)] \{\Delta\} = \{P\}$$

The nature of nonlinear problems is such that no practical numerical technique exists which renders the solution routine. Every nonlinear problem encountered requires special, problem-oriented attention to select a "best" technique and determine its range of applicability and sensitivity with respect to important problem parameters.

In the light of these comments, it is pertinent to record the alternative approaches which were examined within the scope of the present study. The most obvious iterative

form, and the first attempted, is written

$$\{\Delta\}_{q+1} = \left[[K] + [N1(\Delta_q)] + [N2(\Delta_q^2)] \right]^{-1} \{P\}$$

The apparent shortcoming of this form is the singularity of the augmented stiffness matrix at the point of instability. This short coming might be tolerated if favorable convergence characteristics were exhibited at load levels less than the critical load; however, this is not the case.

The second iteration scheme applied to the set of nonlinear equations was an over/under relaxation stated as

$$\{\Delta\}_{q+1} = \{\Delta\}_q + \textcircled{H} \left\{ \{\tilde{\Delta}\}_{q+1} - \{\Delta\}_q \right\}$$

where

$$\{\tilde{\Delta}\}_{q+1} = [K]^{-1} \left\{ \{P\} - [N1(\Delta_q)] \{\Delta\}_q - [N2(\Delta_q^2)] \{\Delta\}_q \right\}$$

The quantity \textcircled{H} is the scalar relaxation factor. This approach was also abandoned in the absence of satisfactory convergence characteristics.

At this point in the study the approach adopted was that of a direct attack on the potential energy function by the application of mathematical programming methods. The intense research activity in this area has achieved a high degree of success. The subject nonlinear structural analysis method is an additional benefit of this mathematical programming research. Selection of a particular method was discussed in Section III.

The final discussion topic listed at the outset regards the effectiveness of the computer program as a working tool for the analysis of geometrically nonlinear structures. The definitized analytical and computational procedures and the applications reported herein indicate a significant stride in this direction. However, the analysis capability in its present form, falls short of the more ambitious goal of a practical working tool for the analysis of large scale nonlinear structures.

The mathematical programming or potential energy minimization approach developed remains attractive relative to available alternatives. Refinement in computational aspects is needed to improve effectiveness and efficiency of numerical solution. The payoff of such a refinement is potentially great. Certainly, the need for a practical nonlinear structural analysis method exists.

Several specific modifications which promise significant impact have been brought



to light by experience in application accumulated in the present study. Firstly, improved performance could be achieved by further breakdown of the computer program organization into module units.

The volume of numerical calculation within an iteration cycle and the number of iterations required is such as to warrant the sacrifice of programming simplicity in order to minimize redundant calculation. The strategy here is one of saving results of previous calculation, and to be workable, requires the availability of fast-access peripheral storage equipment.

A second suggested modification straddles the interface between program organization and minimization technique. Recalling that the whole of the structure potential energy and gradient at a point are assembled from individual element contributions, it follows that certain minimization steps (Equation 3-1) could be considered which would require only fractional recalculation of energy and gradient values. That is, energy and gradient values could be very efficiently "corrected" rather than completely recalculated.

A third, and most expedient, modification likely to improve efficiency is to bring a highly directed square metric numerical minimization technique to bear on the problem. In virtue of the modular organization of the computer program, this is easily accomplished. The value of this approach is limited by the need to store and manipulate dense matrices of large order.

In general, the success achieved supports the technical approach developed and the difficulties encountered reaffirm the complexity of the general nonlinear analysis problem. The development of practical working tools for nonlinear structural analysis is expected to remain an evolutionary process.



VI. CONCLUSIONS AND RECOMMENDATIONS

The following abbreviated list summarizes the extended discussion presented in Section V. Based on the investigation conducted, it is concluded that:

1. The finite displacement analyses conducted together with the computer program and data delivered provide the basis for confirming the integrity of the Apollo aft heat shield under water impact loading conditions.
2. The geometrically nonlinear analysis capability developed has a high potential for application to structural components of future NASA programs.
3. Further development is required to improve effectiveness and efficiency in order to render the analysis capability a practical working tool for nonlinear structural analysis. Implementation of a highly directed minimization technique, (Reference 21) is recommended.



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